
A Physiologically Motivated Analytic Model for Optimal Bicycle Gearing

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Abstract

The performance of a cyclist can be optimized by selecting the most appropriate gear ratio for a rider's physiological profile, bike-rider setup, and environmental conditions. Currently, there is no analytic model available to guide gear ratio selection, and the aim of this study is to fill this gap. We study the kinematics of a cyclist using a physiologically motivated human power model. Analytic expressions for the position and velocity of the cyclist as functions of time are presented. We show how these formulas can be used to optimize the choice of gear ratio, taking into account the transient behavior of sprint-like efforts. The model can be used to guide gear ratio selection in a variety of cycling disciplines.

Keywords

cycling, human-powered vehicles, aerodynamic drag, gear selection

Introduction

Models describing the kinematics of a cyclist, which date back to the 1894 work of Bourlet (see Malizia and Blocken¹), are valuable for understanding how

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specific parameters of the bike-rider setup influence a rider's performance under various environmental conditions. As an example, consider the work in Knight², which resolves a well-known paradox, namely that the time saved during a fast descent will not compensate for time lost from climbing the same hill. Similarly, it was shown that a tailwind will not compensate for a headwind on an out-and-back ride. Both of these paradoxes stem from the quadratic nature of the drag force in high-Reynolds number flows.

The model used in Knight² falls into a common class that considers a steady-state cyclist where the rider's power output balances the net resistive forces. With the rider's acceleration set to zero, there is no transient behavior included in the analysis. Regardless, these models have been successfully validated³ and used in numerous optimization and pacing studies.⁴⁻⁸ We note that while some of these references do include a term for the change in kinetic energy, as far as the author is aware, the studies themselves consider nearly steady-state efforts.

A study by Bier⁹ of "Everesting"—repeatedly climbing and descending the same hill until acquiring the elevation of Mount Everest—did necessitate the inclusion of some transient effects, namely finding the elapsed time before the terminal velocity is reached during a descent. In such a circumstance, it is only necessary to consider a freewheeling (non-pedaling) cyclist where an analytic solution was obtained. It would clearly be advantageous if analytic solutions could be developed with fewer restrictions.

In track time trials, riders try to obtain the fastest time over a specified race distance (often 1 km for men and 500 m for women) starting from a standstill. Because a fixed gear bike is required for all track time trials, choosing the correct gear ratio is paramount for race performance. A lower gear ratio will allow for rapid acceleration but comes at the expense of a lower top speed. The optimal gear choice will balance these two effects and ultimately depend on the rider's physiology. The main goal of this work is to provide an analytic and quantitative model to guide gear selection for such events.

This work develops an analytic solution for the kinematics of a cyclist using a physiologically motivated power function for the rider. The kinematic equations that result are relatively compact and can be easily plotted as they make use of no special functions. As an example of the utility of having such a solution available, we show how to optimize for gear ratio given different bike-rider setups (e.g. mass, drag, rolling resistance), road grades and race distances. The resulting equations for calculating an optimal gear ratio constitute the main contribution of this work.

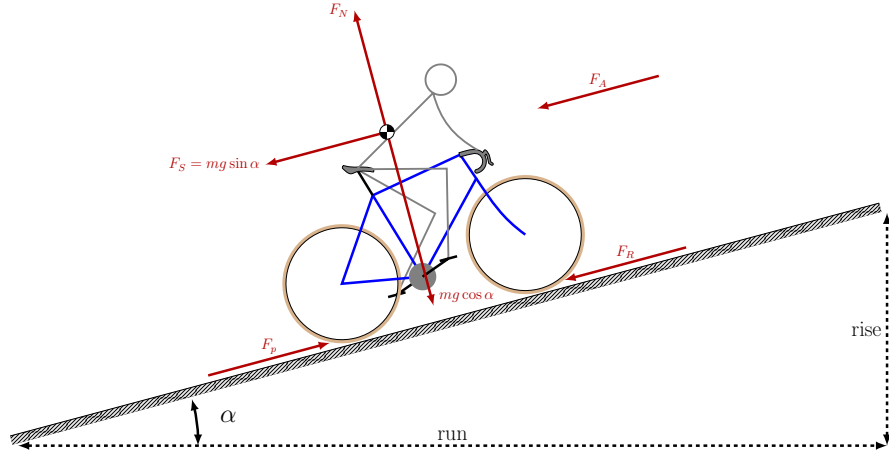


Figure 1. Forces acting on the bike-rider system. F_A is the aerodynamic drag, F_R is the rolling resistance, F_S is the component of the force due to gravity along the slope, F_N is the normal force, and F_p represents the propulsive force. The magnitudes of the forces are not drawn to scale.

Methods

We consider the scenario where a bicyclist rides on a smooth straight road as depicted in Figure 1. We follow the setup and notation of chapter 5 in Wilson et al.¹⁰, where the balance of forces on the bike-rider system is expressed as,

$$F_p = F_A + F_S + F_R + m\ddot{x}. \quad (1)$$

In the above expression, F_p is the propulsive force on the bicycle's rear tire, F_A is the aerodynamic drag force, F_S is the slope resistance from gravity, F_R is the rolling resistance, m is the mass of the bike-rider system and $\ddot{x} = \dot{v} = a$ is the acceleration of the bike-rider system.

We now discuss how we model these forces, starting with F_S , the force of gravity acting along the direction of the slope. It is expressed as,

$$F_S = mg \sin \alpha, \quad (2)$$

where m is the combined mass of the bike and rider and g is the acceleration due to gravity. The angle α is the slope angle, $\alpha = \arctan\left(\frac{\text{rise}}{\text{run}}\right)$, typically reported as the road's grade in percent, $\% \text{ grade} = 100 \times \text{rise/run}$. A positive (negative)

grade represents a rider ascending (descending). Figure 1 is drawn using a slope angle of 15° , a very steep grade of $\approx 26.8\%$.

The rolling resistance, F_R , is expressed as a dimensionless coefficient of rolling resistance, C_{rr} , times the normal force,

$$F_R = C_{rr}F_N = C_{rr}mg \cos \alpha. \quad (3)$$

Because the slope resistance and rolling resistance both produce a constant acceleration, we can simplify the discussion that follows by combining these terms into a single effective acceleration constant, g_s ,

$$g_s \equiv C_{rr}g \cos \alpha + g \sin \alpha. \quad (4)$$

When riding up an incline, g_s is positive. If going down an incline, g_s will be negative when the force of gravity surpasses the rolling resistance. The aerodynamic drag, F_A , is expressed as,

$$F_A = \frac{1}{2}\rho(C_DA)v^2, \quad (5)$$

where ρ is the air density, C_D is a dimensionless drag constant, and A is the projected frontal surface area¹¹ of the bike-rider system. Because this latter quantity is difficult to measure accurately (although image analysis algorithms¹² have made progress in this direction), a combined metric known as the drag area (C_DA) is typically reported in units of m^2 . To simplify the equations that follow, we introduce an air-drag constant c_d , defined as,

$$c_d \equiv \frac{1}{2m}\rho C_DA, \quad \text{such that} \quad \frac{F_A}{m} = c_d v^2. \quad (6)$$

The inverse of c_d physically represents a characteristic distance scale over which the aerodynamic drag decelerates the cyclist. Consider a cyclist with some initial velocity, v_0 . In the absence of any other forces, aerodynamic drag will reduce the bike's speed to half its starting value in a time $t_{1/2} = 1/(c_d v_0)$ over a distance of $x_{1/2} = \log 2/c_d$.

The final term to discuss is the propulsive force appearing on the left-hand side of Eq. 1. The power delivered to the rear tire equals ηP_F where η accounts for the drivetrain transmission efficiency. The propulsive force, F_p , at the rear

tire is therefore related to the rider's power output through the expression,

$$F_p = \frac{\eta P_F}{v}. \quad (7)$$

Using the above expression along with our newly defined parameters g_s and c_d , the force balance Eq. 1 can be recast as,

$$\frac{\eta P_F}{mv} = c_d v^2 + g_s + \dot{v}. \quad (8)$$

In this work, c_d and g_s are both treated as constants. Future work could generalize these results to include velocity-dependent coefficients as supported by one such coast-down model.¹³

We will discuss the human power output model and present the functional form of P_F in the next section. To produce quantitative results we will consider three case studies, each with distinct parameter choices as listed in Table 1. We stress that these parameter sets are not intended to provide universally optimal gearing recommendations but serve as a means to demonstrate the analytical framework. The parameters for our “Touring” profile are taken from the “Road bike, touring position” of Table 5.1 in Wilson et al.¹⁰ The “Racing” profile is taken from the “Racing bike, rider crouched, tight clothing” of Table 5.1 in Wilson et al.¹⁰ We have set the drivetrain efficiency to 100% for these two profiles, noting that all expressions that follow contain the combined product of $\eta P_{\max}$. Finally, the “Sprinting” profile uses the values determined in Clanet et al.¹⁴ Because $P_{\max}$ and PR_{opt} are specific to our model we list values representative of what is reported in Clanet et al.¹⁴

The cyclist power model

As anyone who rides a bike will immediately recognize, there is a comfortable pedaling rate or cadence that one reflexively settles into. This cadence will depend on how hard the rider is working; when pushing high gears during a climb, the pedaling rate will be markedly slower. On the opposite end of the spectrum, trying to accelerate by simply spinning faster, as opposed to shifting gears, will be ineffective.

The mechanical performance of a muscle is often quantified by the speed at which a muscle shortens under the influence of an external load. The functional form of this relationship between muscle velocity and load is successfully

		Touring	Racing	Sprinting
Acceleration of gravity (m/s ²)	g	9.81	9.81	9.81
Air density (kg/m ³)	ρ	1.225	1.225	1.140
Rider + bike mass (kg)	m	95	81	100
Drag area (m ²)	$(C_D A)$	0.4	0.32	0.21
Rolling-resistance coefficient	C_{rr}	0.0045	0.0030	0.0025
Air-drag constant (m ⁻¹)	c_d	0.0026	0.0024	0.0012
Effective acceleration (m/s ²)	g_s	0.0441	0.0294	0.0245
Drivetrain efficiency	η	1	1	0.91
Optimum pedaling rate (rpm)	PR_{opt}	80	110	130
Maximum power at PR_{opt} (W)	P_{max}	200	900	1800
Wheel circumference (m)		2.146	2.096	2.099

Table 1. The columns labeled Touring and Racing use the parameters from Table 5.1 of Wilson et al. ¹⁰ for a “Road bike, touring position” and “Racing bike, rider crouched, tight clothing”, respectively. The Sprinting profile uses the values from Clanet et al. ¹⁴ with baseline values of P_{max} and PR_{opt} taken from the ergocycle fits, whereas Figure 7 uses distance-specific values from sprint data. The values of c_d and g_s are calculated from Eqs. 6 and 4 using a grade of 0%.

described by Hill’s equation ¹⁵, a hyperbolic relation between these quantities. While Hill’s equation may be empirically successful for muscles activating on an individual basis, the dynamics of cycling is more complex. Studies going as far back as 1928 ¹⁶ have found a linear relationship between force and velocity during cycling. See Driss and Vandewalle ¹⁷ for a comprehensive review along with Alcazar et al. ¹⁸ for a discussion of the detailed shape of the force-velocity curve and Douglas et al. ¹⁹ for a review addressing the physiological limitations of these profiles in sprint cycling.

This work will focus solely on a linear force-velocity curve such that the torque supplied by a cyclist decreases linearly with pedaling rate, torque = $C_1 - C_2 \cdot PR$. The power output is proportional to the torque times the pedaling rate, resulting in Power $\propto C_1 \cdot PR - C_2 \cdot PR^2$. This functional form follows from the simple fact that there is some optimal pedaling rate around which one can perform a Taylor series expansion.

Specifically, we take the rider’s power (foot power, P_F) as a quadratic function of their pedaling rate (PR),

$$P_F = P_{max} \left[2 \left(\frac{PR}{PR_{opt}} \right) - \left(\frac{PR}{PR_{opt}} \right)^2 \right], \quad (9)$$

where $P_{\max}$ is the rider's maximum output power at their optimum pedaling rate PR_{opt} . This functional form has been validated in many studies.^{20–22}

This work will treat the two parameters, $P_{\max}$ and PR_{opt} , as constants that describe the overall fitness level of the cyclist. The reality is that these two parameters will vary with exercise intensity²³ and riding position.²⁴ We therefore expect the values of $P_{\max}$ and PR_{opt} that we adopt for our three case studies to be different. The torque/power versus pedaling rate curves are shown in Figure 2 for our Touring and Racing profiles. The torque is given as $60P_F/(2\pi\text{PR})$ with P_F from Eq. 9.

To incorporate the rider's power curve, Eq. 9, into the force balance Eq. 8, we recognize that the pedaling rate is a function of the bike's velocity, governed by the gearing and tire size. We use development, defined as the distance in meters traveled by the bike under one full rotation of the crank, as our preferred measure of gearing,

$$D = \text{Wheel Circumference} \times \frac{\text{Front Chainring Teeth}}{\text{Rear Sprocket Teeth}}. \quad (10)$$

Throughout our discussion, we will refer to the gear ratio which is the number of front chainring teeth divided by the number of rear sprocket teeth. It can be found by dividing the development D by the wheel circumference.

The bike velocity, v , is then related to the pedaling rate through the relation,

$$v = \frac{D}{60}\text{PR}, \quad v_{\text{opt}} = \frac{D}{60}\text{PR}_{\text{opt}}. \quad (11)$$

We define the constant v_{opt} as the bike's speed when pedaling at the optimum cadence. Expressed in these variables, the foot power delivered to the bike is,

$$P_F = P_{\max} \left[2 \left(\frac{v}{v_{\text{opt}}} \right) - \left(\frac{v}{v_{\text{opt}}} \right)^2 \right]. \quad (12)$$

Substituting this expression into the force balance Eq. 8 results in,

$$\frac{\eta P_{\max}}{m v_{\text{opt}}} \left(2 - \frac{v}{v_{\text{opt}}} \right) = c_d v^2 + g_s + \dot{v}. \quad (13)$$

As all of the results in this paper stem from the above equation, it is worth taking a moment to summarize what it entails. The term on the left-hand side represents the acceleration of the cyclist in the absence of any resistive forces,

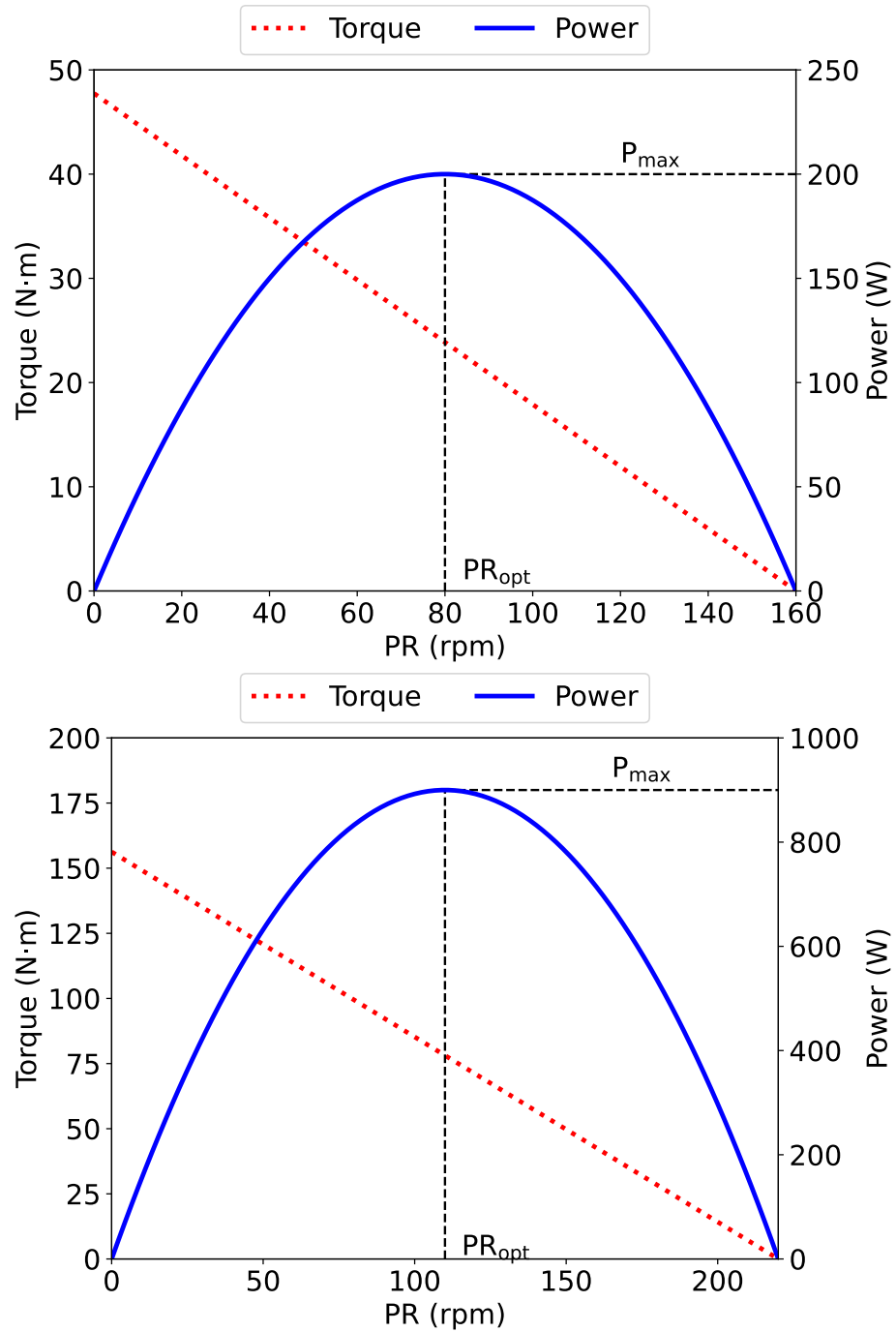


Figure 2. Torque and Power vs pedaling rate for our Touring cyclist (upper) and our Racing cyclist (lower). The left y-axis shows the torque in units of Nm. The right y-axis shows the rider's foot power P_F in Watts. Both are plotted as a function of pedaling rate PR in rpm with the maximum power occurring at PR_{opt} .

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and is proportional to the torque provided by the rider. The torque is a linearly decreasing function of pedaling rate as shown in Figure 2. The first term on the right-hand side is due to aerodynamic drag and the second term on the right-hand side is the effective acceleration due to the combination of gravity and rolling resistance. When the torque supplied by the rider is able to overcome the resistive forces, a net acceleration of the bike-rider system is obtained. Note that Eq. 13 is only valid when $v \leq 2v_{\text{opt}}$.

Results

The analytic solution of Eq. 13 will be one of the main contributions of this work. In the next section we first study the steady-state behavior obtained by setting $\dot{v} = 0$, and find an analytic expression for the optimum gear ratio in such a scenario. The section that follows will present the full time-dependent solutions along with compact expressions for calculating the optimal gear ratio of transient efforts.

Steady-state cycling

We start by studying the case when a rider's power output matches the opposing forces of friction, drag and gravity. In this case, the rider's acceleration $\dot{v} = 0$ and Eq. 13 is quadratic in v . The rider's steady-state velocity, which we call v_{∞} , takes the form,

$$v_{\infty} = \sqrt{\bar{v}^2 + 4\bar{v}v_{\text{opt}} - \frac{g_s}{c_d}} - \bar{v}. \quad (14)$$

We have introduced an auxiliary velocity variable $\bar{v}$ to simplify the expression,

$$\bar{v} \equiv \frac{\eta P_{\text{max}}}{2mc_d v_{\text{opt}}^2}. \quad (15)$$

The quantity $\bar{v}$ represents the velocity at which the rider's power output is exhausted by aerodynamic drag. As will be discussed later on, a larger $\bar{v}$ results in a quicker approach to steady-state velocity.

It is important to note that both $v_{\text{opt}} \propto D$ and $\bar{v} \propto D^{-2}$ are functions of the development, D . Therefore, the rider's steady-state velocity v_{∞} is an algebraic function of D as well.

We wish to optimize v_{∞} assuming a rider's physiological ability, bike and environment (characterized through the parameters $P_{\text{max}}, \text{PR}_{\text{opt}}, c_d, m, g_s$)

remain fixed. By maximizing Eq. 14 with respect to D , we find that the optimum development can be expressed as,

$$D_{\text{opt,ss}} = \frac{60}{\text{PR}_{\text{opt}}} \sqrt{\frac{4g_s}{3c_d}} \sinh \left[\frac{1}{3} \text{arcsinh} \left(\sqrt{\frac{27c_d\eta^2 P_{\text{max}}^2}{4m^2 g_s^3}} \right) \right]. \quad (16)$$

We use the notation, $D_{\text{opt,ss}}$ to remind ourselves that this result is for steady-state efforts. Only when a bike's gearing corresponds to $D_{\text{opt,ss}}$ will a rider be able to achieve the fastest possible speed for their physiological profile. This steady-state velocity is $v_{\infty,ss} = v_{\text{opt,ss}} = D_{\text{opt,ss}} \text{PR}_{\text{opt}}/60$ and occurs when the rider's cadence is PR_{opt} .

The above formula simplifies considerably in the limit of $c_d \rightarrow 0$, which can be used as an approximation when the speed is low enough to neglect aerodynamic drag,

$$D_{\text{opt,ss}} = \frac{60}{\text{PR}_{\text{opt}}} \left(\frac{\eta P_{\text{max}}}{m g_s} \right). \quad (17)$$

In the limit of $g_s = 0$ (valid at higher speeds when aerodynamic drag is the dominant force) we have,

$$D_{\text{opt,ss}} = \frac{60}{\text{PR}_{\text{opt}}} \left(\frac{\eta P_{\text{max}}}{m c_d} \right)^{1/3}. \quad (18)$$

We explore the consequences of riding with sub-optimal gearing by asking what a rider's steady-state velocity, v_{∞} , will be on a bike with a fixed development. We use the parameters for our Touring rider for this aspect of our work.

In Figure 3, we plot the steady-state velocity v_{∞} of Eq. 14 as a function of development D for a flat road (upper plot) and a road with a grade of 2.5% (lower plot). The vertical dashed lines show the location of the optimal development as given by Eq. 16. The points on the curve show the available gearing for a typical touring bike. In both scenarios it is easy to identify a choice of gears that is nearly optimal.

The quantity v_{∞} as plotted in Figure 3 is the steady-state speed a rider will have in a fixed gearing. For example, consider the upper plot for a road with zero grade. If the rider's bike happens to have a gear ratio of 3.05, corresponding to the optimal development of $D = 6.55$ m, the rider's steady-state speed will be $v_{\infty} = 31.451$ kph, the maximum of the curve. The closest gearing we have

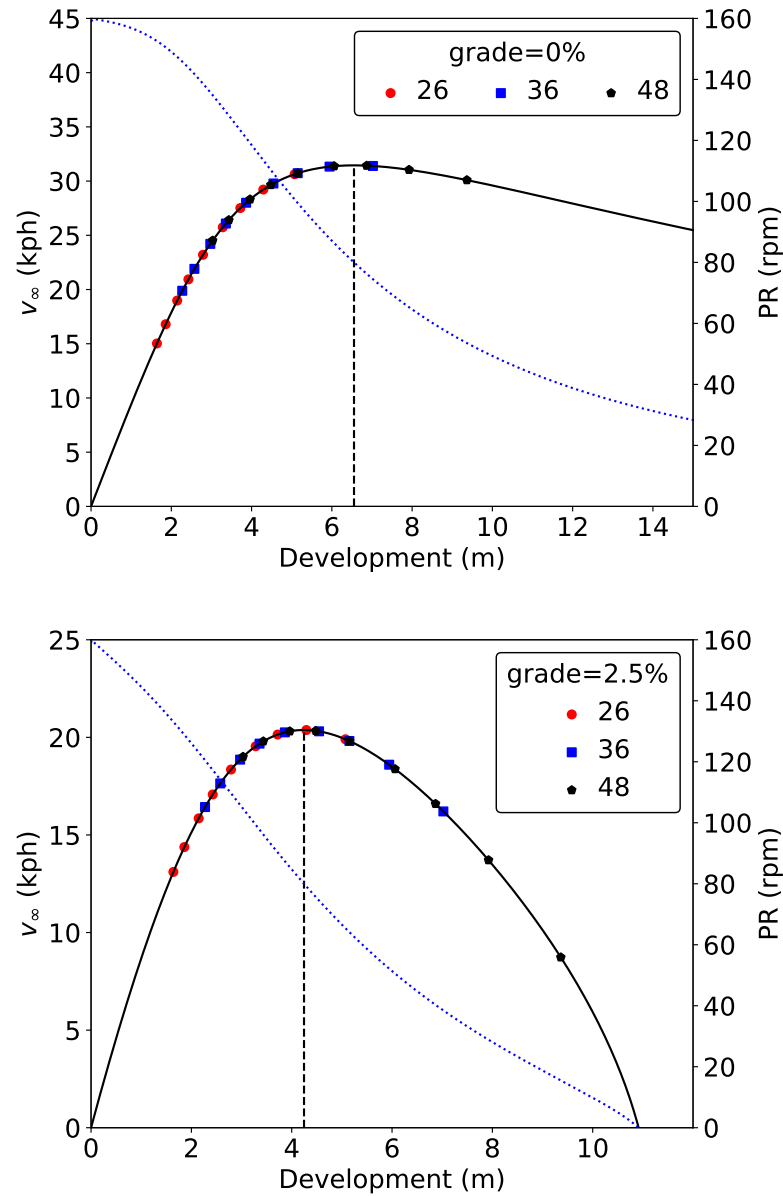


Figure 3. Steady-state velocity in kph versus development in meters for our Touring rider with a grade of 0% (upper) and 2.5% (lower). The points show available developments for a common drivetrain setup. A 26/36/48 front chainring and a rear hub having sprocket sizes 11, 13, 15, 17, 20, 23, 26, 30, 34 (right to left on plot). The dotted blue curve displays the pedaling rate in rpm on the right y-axis.

in this example is a gear ratio of 3.2 corresponding to a chainring/sprocket of 48/15 (third black point from the right). This choice yields a steady-state speed of $v_\infty = 31.425$ kph, and creates a gap of 26 meters over a one-hour effort.

The dashed blue curve in the plot shows the pedaling rate. When riding at the optimal development, the pedaling rate is optimal and the rider's power output is a maximum, 80 rpm and 200 W for our Touring rider. At the slightly less optimal 48/15 gearing we find a pedaling rate of $PR = 76$ rpm and a power output of $P_F = 199.6$ W.

The lower plot of Figure 3 tells a similar story when our Touring rider is on a road with a 2.5% grade. In this case, the optimal development occurs at $D = 4.25$ m with a steady-state speed of $v_\infty = 20.38$ kph. A setup with a gear ratio of 26/13 results in a steady-state speed that is very nearly optimal. Contrast this previous result with the extreme case of the rider being in the inadvisable setup of 48/11 (the rightmost black point), where the resulting steady-state speed would be only $v_\infty = 8.73$ kph.

Time-dependent performance

The force balance Eq. 13 admits an analytic solution for the velocity and position as a function of time and is presented in Eqs. 19 and 20, respectively.

$$v(t) = (v_\infty + \bar{v}) \tanh \left[c_d (v_\infty + \bar{v}) t + \tanh^{-1} \left(\frac{v_0 + \bar{v}}{v_\infty + \bar{v}} \right) \right] - \bar{v} \quad (19)$$

$$x(t) = \frac{1}{2c_d} \ln \left\{ \left(1 - \frac{(v_0 + \bar{v})^2}{(v_\infty + \bar{v})^2} \right) \cosh^2 \left[c_d (v_\infty + \bar{v}) t + \tanh^{-1} \left(\frac{v_0 + \bar{v}}{v_\infty + \bar{v}} \right) \right] \right\} - \bar{v}t \quad (20)$$

These equations are plotted in Figure 4 for two representative gear ratios using the parameters from our Racing profile. We have set the initial velocity, $v_0 = 0$, in what follows. Note that the above setup is valid for $v_0 \leq v_\infty$ only. As any trained cyclist is well aware, the smaller gear ratio is advantageous for quicker acceleration while the larger gear ratio results in a higher speed for endurance efforts. The auxiliary variable $\bar{v}$ is what controls the rate of approach to the steady-state velocity. The red curve (48/16) has a 50% larger $\bar{v}$ than the dashed blue (52/14) curve and the more rapid approach to terminal velocity is readily apparent in the upper plot of Figure 4. This early time

behavior can be quantified by looking at the initial acceleration $a(t=0) = -g_s + 4c_d v_{\text{opt}} \bar{v} = -g_s + 120\eta P_{\text{max}}/\text{PR}_{\text{opt}}(mD)^{-1}$, which increases linearly with the auxiliary variable $\bar{v}$. In terms of the bike-rider parameters, to maximize the initial acceleration, we should try to minimize the total mass and use a low gear ratio.

The lower plot of Figure 4 shows the position as a function of time with the gray horizontal dotted lines drawn at four representative race distances. Inspecting the plots, we find that the lower gear ratio (solid red curve) results in quicker times for the 250 and 500 m efforts. For 750 and 1000 m efforts it is more advantageous to switch to the 52/14 setup (dashed blue curve). Of course, this conclusion is specific to the choice of parameters made in our Racing model.

To solve for the optimum race time, we would need a closed-form solution for $t(x)$, which is unfortunately not possible because Eq. 20 is a transcendental equation. However, we found that a late-time asymptotic expansion,

$$t = \frac{x}{v_{\infty}} + \frac{1}{c_d v_{\infty}} \ln \left[\frac{2(v_{\infty} + \bar{v})}{v_0 + v_{\infty} + 2\bar{v}} \right], \quad (21)$$

worked extremely well for the three cycling profiles used in this work. We now use this expression to demonstrate how changes in gear ratio affect race times. In Figure 5 we have plotted Eq. 21 to show how race time increases as a function of development using four representative race distances of $x = 250, 500, 750, 1000$ m. As expected, the optimum development corresponding to the minimum of each curve increases with race distance. The unlabeled gray curve shows the asymptotic result for an infinitely long race where the transient acceleration phase has little effect on the total race time. The optimal development for this limiting case is given by Eq. 16.

To derive an analytic expression for the optimum development for transient efforts, we must minimize Eq. 21 with respect to D . We note that the dependence of the race time t on D in Eq. 21 is extremely complicated because both v_{∞} and $\bar{v}$ are functions of D . We have managed to find an analytic solution that works well by using a [1/1] Padé approximant valid for large race distances,

$$D_{\text{opt}} = D_{\text{opt,ss}} \left(1 - \frac{k}{c_d x + \tilde{x}_0} \right). \quad (22)$$

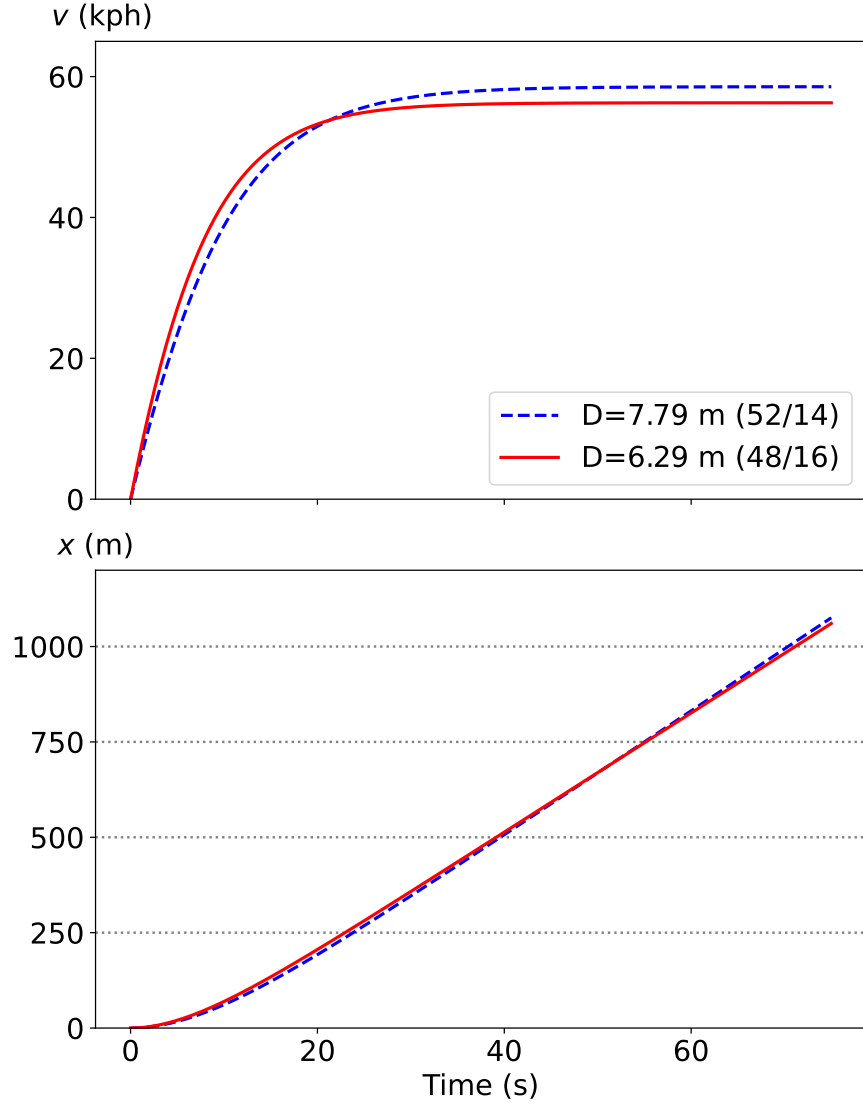


Figure 4. Velocity in kph (top) and position in m (bottom) of our Racing cyclist as a function of time. The solid red curve has a lower gear ratio of $48/16 = 3$ with a resulting development of $D = 6.29$ m. The dashed blue curve is for a higher $52/14 \approx 3.7$ gear ratio and a $D = 7.79$ m.

where $D_{\text{opt,ss}}$ is the optimum development for a steady-state effort as given in Eq. 16. The dimensionless scale factor k and offset $\tilde{x}_0$ are calculated to be,

$$k = \left[1 + \frac{\eta P_{\max}}{m c_d v_{\text{opt,ss}}^3} \right]^{-1}, \quad \tilde{x}_0 = \frac{2k^2(1-k)}{1+k} + \ln(1+k), \quad (23)$$

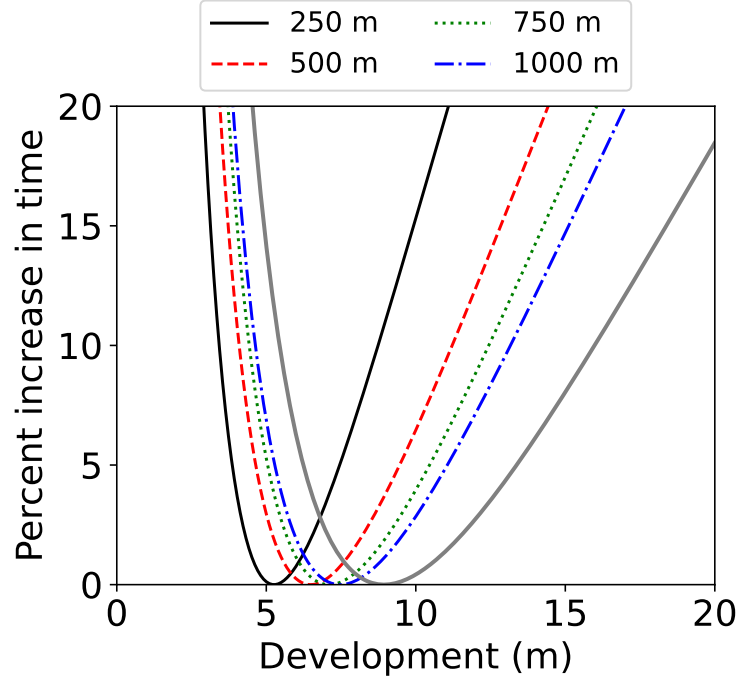


Figure 5. Percent change in race time for our Racing cyclist as a function of development in meters. Results are shown for race distances of 250 (solid black), 500 (dashed red), 750 (dotted green) and 1000 (dot-dashed blue) meters. The unlabeled gray curve shows the asymptotic result for a long race where the initial transient acceleration has little to no consequence.

where $v_{\text{opt,ss}} = D_{\text{opt,ss}} \text{PR}_{\text{opt}}/60$ is the steady-state velocity at optimum pedaling rate and development. We note that the scaling factor k is a function of the ratio of the rider's power to the power consumed by aerodynamic drag. It is exactly this variable that controls how rapidly a rider approaches steady state. As expected, we find that as $k \rightarrow 0$ the optimum development D_{opt} approaches $D_{\text{opt,ss}}$ irrespective of the distance. In Figure 6, we plot Eq. 22 as the solid black curve for our Racing profile.

To test the approximations made in arriving at Eq. 22, we solve for the optimum development numerically using two different approaches. The dot-dashed magenta curve shows the optimum development obtained by numerically minimizing Eq. 21 with respect to D . The dashed gray curve is the exact result found by numerically inverting Eq. 20 and then minimizing the resulting race time with respect to D . These results indicate that the asymptotic expansion

presented in Eq. 21 and the Padé approximant of Eq. 22 work well for race distances larger than about 250 m.

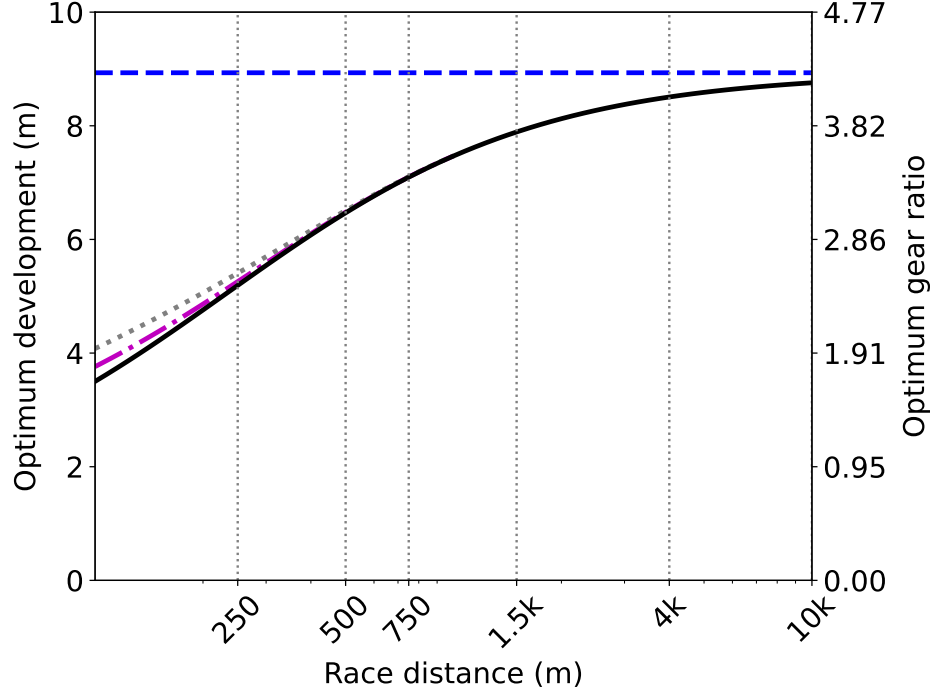


Figure 6. Optimal development in meters for our Racing profile as a function of race distance in meters. The y-axis on the right shows the equivalent gear ratio. The dashed blue horizontal line shows the steady-state optimal development given by Eq. 16. The solid black line shows the Padé approximation given in Eq. 22. The dot-dashed magenta curve shows the result of minimizing Eq. 21. The dashed gray curve shows the exact result as explained in the text.

Discussion

The main goal of this work was to identify analytic expressions to help guide gear selection. For steady-state cycling we derived Eq. 16, which allows a user to find an appropriate gear ratio for endurance efforts where the transient acceleration phase can be ignored. For shorter sprint-like efforts, we derived Eq. 22, which works for race distances larger than 250 m. For shorter efforts, a better approximation can be obtained by minimizing Eq. 21.

We now discuss work that goes beyond some of the simplifying assumptions we have made but which ultimately relies on numerical solutions where the sensitivity to different parameter choices may not be as readily apparent.

Many race efforts take place in a velodrome where the effects of riding on a banked track need to be taken into account. The work of Lukes et al.²⁵ includes the tipping motion of the rider as they navigate a corner with centripetal forces and additional rolling resistance from tire scrubbing—slipping of the front wheel on a banked surface. It has also recently been demonstrated by Clanet et al.²⁶ that there is a lag between when the power is generated at the crank and delivered to the wheel, due to the bike-rider system storing and releasing energy over the crank cycle.

More sophisticated studies^{27,28} have shown that athletes fail to select optimal gear ratios during track sprint events, and while we do not expect the results presented here to provide a quantitative description of a track cyclist, our hope is that our analytic expressions can provide a starting point for better equipment selection.

Another assumption in our framework is that the power model of Eq. 9 is independent of time. During a track cycling event the rider starts in a standing position with a larger power output and lower cadence. After a short time, the rider then settles into a seated position with lower power output for the remainder of the race. This effect is clearly not captured by Eq. 9, which would dictate the opposite behavior. The work of Dunst and Grüneberger²⁹ modeled this effect by separating the race into standing and seated phases. We could envision doing something similar by using different values of $P_{\max}$, PR_{opt} , and c_d for the two phases. On top of these two phases, there is a fatigue-induced reduction in power output that was successfully modeled by an exponential.²⁹ Unfortunately, we were unable to find an analytic expression when introducing such a time-dependence in our framework.

To substantiate our model, we attempt to compare our results to the published sprint data from Clanet et al.¹⁴, which is publicly available from Clanet et al.³⁰ In Clanet et al.¹⁴, the authors developed a model where the fatigue-induced reduction in power depends on the time spent above a rider's critical power. This model successfully described their experimental data collected from an elite sprinter on an ergocycle and in a velodrome. We reproduce in Figure 7 the experimental data points and the results from their effective time theory (dotted curves) for race distances of 250 m, 500 m, 750 m and 1000 m.

To compare our analytic treatment with the data, we need to carefully consider how to choose the parameters $P_{\max}$ and PR_{opt} . The values of $P_{\max}$ and PR_{opt} reported in the sprinter profile in Table 1 are acquired from 3–5 second sprints on an ergocycle, which may not be representative of the sprinter’s performance in actual racing events.

To choose representative values for $P_{\max}$ and PR_{opt} , we look at the data for the 250 m and 1000 m efforts. For the 250 m effort, a majority of the time is spent in the standing position. During this phase the maximum power output is 2015.8 W at a pedaling rate of 105.3 rpm. For the 1000 m effort, a majority of the time is spent in the seated position. During the seated phase, the maximum power output is 1448.1 W at a pedaling rate of 106.2 rpm. For our purposes, we fix $\text{PR}_{\text{opt}} = 105$ rpm for all race distances and take $P_{\max} = 2016$ W for the 250 m effort and $P_{\max} = 1448$ W for the 1000 m effort. For the other race distances we choose $P_{\max}$ values that linearly interpolate between the 250 m and 1000 m values.

With these caveats in mind, we now show in Figure 7 the results of Eq. 21 as the solid curves. Each curve was generated using the “Sprinting” parameters from Table 1, with the exception of $\text{PR}_{\text{opt}} = 105$ rpm and $P_{\max}$ values as listed in the figure.

The thickness of the curves represent the variance in drag area between the values of $(C_D A) = 0.21 \text{ m}^2$ for a seated rider and $(C_D A) = 0.27 \text{ m}^2$ for a standing rider, values taken from the wind tunnel tests of Clanet et al.¹⁴ We find fair agreement between the results of Eq. 21 and the data of Clanet et al.¹⁴, which substantiates the use of such a model for gear ratio selection. Given the limitations of the model, we stress that the above analysis is intended only as a qualitative validation for our analytic framework.

We hope that our model can be useful in advancing other studies. For example, a series of works has modeled the Tour de France rather successfully.^{31–33} However, as Ramsey and Goff³⁴ point out, there was considerable uncertainty in modeling the cyclist’s power output as a function of road grade. This variation could be naturally accounted for in our framework. Instead of identifying a power output for each grade, one would identify a gear ratio for each grade with the resulting power output, cadence, and speed given by our model. With fewer parameters, it would be interesting to see if such a model could describe real-world data equally well. Another interesting avenue could be to examine if the analytic formula presented here can provide gearing strategies for Bicycle

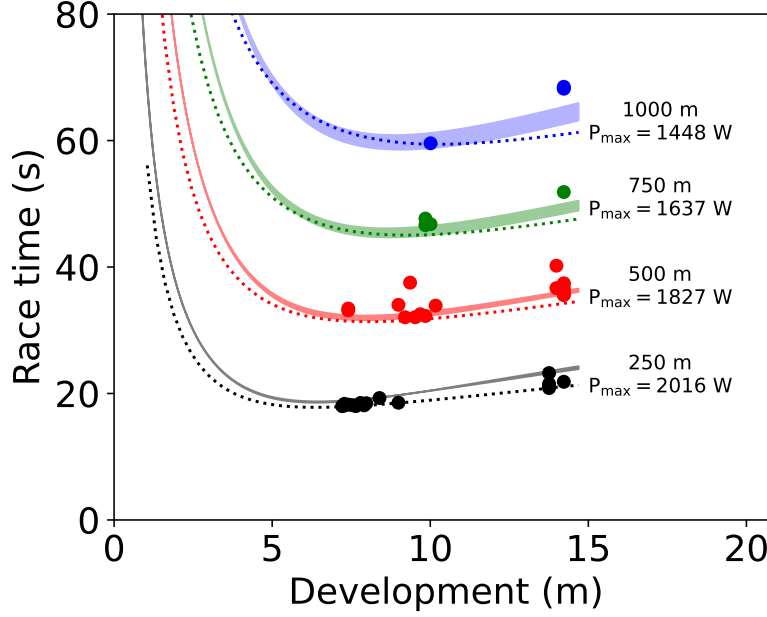


Figure 7. Race times for an elite sprinter as a function of development in meters. Results are shown for race distances of 250 (black), 500 (red), 750 (green) and 1000 (blue) meters. The dotted curves are the effective time theory of ref. Clanet et al.¹⁴ along with the experimental data points collected from that same work. The solid curves are the results of Eq. 21 using the “Sprinting” parameter set from Table 1. The value of $P_{\max}$ was adjusted to the race distance with the values shown. We have taken $PR_{\text{opt}} = 105$ rpm for all race distances as explained in the text.

Moto-Cross (BMX) racing that are in qualitative agreement with the numerical study and measurements from Fortier et al.³⁵

Finally, we note that our framework can also be extended to optimize e-bike performance. If we consider an idealized motor that generates a constant torque τ at the crank, the resulting kinematic Eqs. 19 and 20 remain valid if we redefine,

$$v_{\infty} = \sqrt{\bar{v}^2 + 4\bar{v}v_{\text{opt}} - \frac{g_s}{c_d} + \frac{2\pi\eta\tau}{c_d m D}} - \bar{v}. \quad (24)$$

One can envision incorporating the above solutions into the architecture of control systems³⁶ for e-bikes. We leave a detailed analysis of e-bike optimization to a future study.

Limitations

Throughout this work, we have mentioned in passing many of the restrictions that were imposed to arrive at analytic expressions. We enumerate these limitations here for a better understanding of the intended scope of the model.

1. The physiological parameters $P_{\max}$ and PR_{opt} are treated as constants and cannot account for the fatigue-induced reduction in power output over the course of an effort.
2. Any standing-to-seated transition—which alters $P_{\max}$, PR_{opt} , and the drag area C_DA —is not captured by the time-independent power model.
3. The linear force-velocity curve adopted here is a simplification of a rider’s true physiological performance, particularly when the rider is operating far from PR_{opt} .
4. We have assumed a straight and bump-free road.
5. The model assumes still air and does not account for headwinds, tailwinds, or crosswinds. Aerodynamic interactions from nearby riders are also excluded.
6. The rotational inertia of the wheels is neglected, causing the model to underestimate the inertial resistance during acceleration.
7. Velodrome banking and the associated tire scrubbing are excluded.
8. Speed-independent coefficients are assumed for both rolling resistance and aerodynamic drag. Drivetrain efficiency is assumed independent of chain-line and sprocket/chainring sizes.
9. The analytic approximations we develop are valid for race distances that are longer than approximately 250 m.

Conclusions

In conclusion, an analytic model for the kinematics of a bike-rider system is derived using a physiologically motivated power function for the rider. We have obtained closed-form expressions for the position and velocity as functions of time. We have demonstrated how our results can be used to optimize gear selection and determine the outcomes of sprint or endurance efforts on a straight path or track.

Declaration of conflicting interests

The author declares no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Data availability statement

The code used to produce the figures in this manuscript is publicly available on Zenodo at <https://doi.org/10.5281/zenodo.21791167>.

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